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DECOMPOSING TOPOLOGICAL SPACES INTO TWO RIGID HOMEOMORPHIC SUBSPACES

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ABSTRACT

We prove that $\mathfrak{R}$ can be split into two homeomorphic parts each of which has no autohomeomorphism except the identity. Moreover, this holds for many separable normed vector spaces over the rationals.

MAIN THEOREM. *We say that a topological space in X can be* rigidly halved *if X can be partitioned to two homeomorphic rigid sets.*

Let X be a separable normed vector space of power $2^{\aleph_0}$ over the field $\mathbf{Q}$ of all rationals. X can be rigidly halved if it satisfies one of the following conditions:

(1) X has a complete direction, i.e., there is a $z \in X$, $z \neq 0$, such that $\mathfrak{R}z \subseteq X$, where $\mathfrak{R}$ is the set of all real numbers and for every $z \in X$ and $r \in \mathfrak{R}$, rz is defined in the completion X^C of X.

(2) X is meager.

(3) X has an autohomeomorphism of order 2 without fixed points.

COROLLARY. *Let X be a separable normed vector space of power $2^{\aleph_0}$ over Q. X can be rigidly halved if it satisfies one of the following conditions*:

(4) *X is complete.*

(5) *X has a nonvoid bounded clopen set.*

(6) *X has a nontrivial autohomeomorphism which is the identity outside some bounded set.*

HISTORY. In the 1982 North-Holland Calendar, van Mill asked whether $\mathfrak{R}$ can be rigidly halved. Van Mill and Wattel have proved in [3] that the circle

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and some related spaces can be rigidly halved. Van Engelen [2] and the author [1] solved, independently, van Mill's question. The present Main Theorem extends this result to many separable normed vector spaces over **Q**.

We shall now proceed to prove the Main Theorem and then we shall show how the Corollary follows from it.

NOTATION. We denote by **Z** the set of all integers. For a metric space $\langle X, d\rangle$ and $A, B \subseteq X$ we define $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

THEOREM 1. (A) *Let X be a Hausdorff space which satisfies the second countability axiom, and let f, D_1, D_2, E_1, E_2 be as in the following* (1)–(7).

(1) *$E_1 \subseteq D_1$, $E_2 \subseteq D_2$, and D_1, D_2 are disjoint subsets of X.*

(2) *f is an homeomorphism of $X - E_2$ onto $X - E_1$.*

(3) *f maps D_1 onto D_2 and $D_2 - E_2$ onto $D_1 - E_1$.* (Notice that by (1) and (2) D_1, $D_2 - E_2 \subseteq \mathrm{Dom}\, f$, D_2, $D_1 - E_1 \subseteq \mathrm{Range}\, f$.)

(4) *For every $x \in X$ and $n \in \mathbf{Z}$, if $f^{2n+1}(x)$ is defined then $x \neq f^{2n+1}(x)$.*

(5) *For every nonvoid open set U in X,*

$$|U - (D_1 \cup D_2)| = 2^{\aleph_0}.$$

(6) *Every nonvoid open set U in X has a nonvoid open subset V such that for each $n \in \mathbf{Z}$, $d(f^n(V), \bigcup_{m \in \mathbf{Z}, m \neq n} f^m(V)) > 0$.*

(7) *For every $n \in \mathbf{Z}$, $n > 0$, and for every nonvoid open set U, there are $x_i \in U \cap D_1 \cap \mathrm{Dom}\, f^{2n}$, $i < \omega$, such that the sequence $\langle x_i : i < \omega\rangle$ converges to some point $x_\omega \in D_1$ but $\langle f^{2n}(x_i) : i < \omega\rangle$ does not converge to any point in $X - D_2$.*

Then X can be rigidly halved.

(B) *The conclusion of* (A) *holds also if we replace requirements* (5) *and* (6) *by*:
(6)$^{\mathrm{B}}$ *$X - (D_1 \cup D_2)$ is nowhere meager, even if we delete from it* $< 2^{\aleph_0}$ points.

(C) *The conclusion of* (A) *holds also if we replace requirements* (6) *and* (7) *by*:
(6)$^{\mathrm{C}}$ *For every $x \notin D_1 \cup D_2$, $f^2(x) = x$.*

PROOF OF THEOREM 1 — FIRST STAGE. In the first two parts of the proof we assume only (1)–(4) and we set up a procedure for splitting X into subsets A and B such that $A \supseteq D_1 \supseteq E_1$, $B \supseteq D_2 \supseteq E_2$, the given function f is the required homeomorphism of A onto B and f also maps $B - E_2$ onto $A - E_1$. The procedure which we shall set up is sufficiently flexible to allow carrying out steps which will ensure the fulfillment of additional requirements. We shall also assume that, as a result of these additional steps, A is dense in X.

By (1)–(3) we can partition X into sets of the following four types:

Type (i). $\{x, f(x), f^2(x), \dots\}$, where $x \in E_1$;

Type (ii). $\{\dots, f^{-2}(x), f^{-1}(x), x\}$, where $x \in E_2$;

Type (iii). $\{x, f(x), f^2(x), \dots, f^m(x)\}$, where $x \in E_1, f^m(x) \in E_2$;

Type (iv). $\{\dots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \dots\}$, where this set is disjoint from $E_1 \cup E_2$.

Any of the sequences (iv) may contain repetitions, but by (4) its members in the even places are different from those in the odd places.

By (1)–(3) we have, for $n < \omega$, for the sets of type (i) and (iii) $f^{2n}(x) \in D_1$, $f^{2n+1}(x) \in D_2$; for sets of type (ii) $f^{-(2n+1)}(x) \in D_1, f^{-2n}(x) \in D_2$, and for sets of type (iv) either the set is included in $D_1 \cup D_2$, with the members alternating between D_1 and D_2, or else the set is disjoint from $D_1 \cup D_2$.

We shall now construct two ascending sequences $\langle A_\alpha : \alpha < 2^{\aleph_0} \rangle$ and $\langle B_\alpha : \alpha < 2^{\aleph_0} \rangle$ of subsets of X such that for all $\alpha < 2^{\aleph_0}$:

(a) $A_0 = D_1$, $B_0 = D_2$.

(b) $A_\alpha \cup B_\alpha - A_0 \cup B_0$ is a union of sets of type (iv).

(c) If α is a limit ordinal then $A_\alpha = \bigcup_{\beta < \alpha} A_\beta$, $B_\alpha = \bigcup_{\beta < \alpha} B_\beta$.

(d) $A_\alpha \cap B_\alpha = \emptyset$.

(e) f maps A_α onto B_α and $B_\alpha - E_2$ onto $A_\alpha - E_1$.

(f) $|A_\alpha - D_1|$, $|B_\alpha - D_2| \leqq |\alpha| + \aleph_0$.

(g) $\bigcup_{\alpha < 2^{\aleph_0}} (A_\alpha \cup B_\alpha) = X$.

We take $A = \bigcup_{\alpha < 2^{\aleph_0}} A_\alpha$, $B = \bigcup_{\alpha < 2^{\aleph_0}} B_\alpha$, then $A \supseteq A_0 = D_1$, $B \supseteq B_0 = D_2$, A and B are disjoint by (d), $A \cup B = X$ by (g) and, by (e), f maps A onto B and $B - E_2$ on $A - E_1$.

We shall define A_α and B_α by recursion. A_0 and B_0 are given by (a); for a limit ordinal α, A_α and B_α are given by (c). Given A_α and B_α we construct $A_{\alpha+1}$ and $B_{\alpha+1}$ as follows. We take for $A_{\alpha+1} \cup B_{\alpha+1} - A_\alpha \cup B_\alpha$ the union of any number $\leqq \aleph_0$ of sets of type (iv), putting for each such sequence the members in the even places in one of the sets $A_{\alpha+1}$, $B_{\alpha+1}$ and the members in the odd places in the other one. It follows now easily, by induction on α, that requirements (b), (d)–(f) are satisfied. We shall refer to the step of going from A_α, B_α to $A_{\alpha+1}$, $B_{\alpha+1}$ as the *recursion step*. We have considerable freedom in what we can do in the $2^{\aleph_0}$ recursion steps, and what we shall do in them will determine the properties of the sets A and B.

By the second countability axiom $|X| \leqq 2^{\aleph_0}$; let $X = \{x_\beta : \beta < 2^{\aleph_0}\}$. Let us denote with P_β the task of making sure that x_β is in $A_{\alpha+1} \cup B_{\alpha+1}$. P_β is carried out in a recursion step as follows. If $x_\beta \in A_\alpha \cup B_\alpha$ then nothing is done. If $x_\beta \notin A_\alpha \cup B_\alpha$ then, by (a) and (3), x_β is in some set of type (iv); in passing to

$A_{\alpha+1}$, $B_{\alpha+1}$ we add this set to $A_\alpha \cup B_\alpha$. In the $2^{\aleph_0}$ recursion steps we carry out, among other tasks, the tasks P_β, $\beta < 2^{\aleph_0}$, and hence also requirement (g) holds.

Before going on to the next step we need the following lemma, which will enable us to carry out in the recursion steps a diagonalization over all autohomeomorphisms of A.

LEMMA 2. *Let X be a Hausdorff space which satisfies the second countability axiom. Then there is a sequence $\langle g_\alpha : \alpha < 2^{\aleph_0} \rangle$ of one-one functions from subsets of X into X such that for every autohomeomorphism g of a dense subset A of X there is an $\alpha < 2^{\aleph_0}$ such that $g \subseteq g_\alpha$.*

PROOF OF THE LEMMA. Let $\langle U_n : n < \omega \rangle$ be a basis for the topology of X. Let g be an autohomeomorphism of a set A dense in X. We define now a double sequence $\langle V_n, W_n : n < \omega \rangle$ as follows. For each $n < \omega$ we set $V_{2n} = W_{2n+1} = U_n$. For each $n < \omega$ we take for W_{2n} some open set of X such that g maps $A \cap U_n$ into W_{2n} and $A - U_n$ into $X - W_{2n}$ (there is such a set, since g is a homeomorphism and the topology on A is induced by that of X). Similarly we take V_{2n+1} to be an open set such that g^{-1} maps $A \cap U_n$ into V_{2n+1} and $A - U_n$ into $X - V_{2n+1}$. Thus for all $n < \omega$ the one-one function g maps $A \cap V_n$ onto $A \cap W_n$. Therefore, for all $m, n < \omega$

$$(A \cap V_n) \cap (A \cap V_m) = \varnothing \qquad \text{iff} \, (A \cap W_n) \cap (A \cap W_m) = \varnothing .$$

Since A is dense in X, $(A \cap V_n) \cap (A \cap V_m) = \varnothing$ iff $V_n \cap V_m = \varnothing$, and $(A \cap W_n) \cap (A \cap W_m) = \varnothing$ iff $W_n \cap W_m = \varnothing$. As a consequence $V_n \cap V_m = \varnothing$ iff $W_n \cap W_m = \varnothing$.

We say that a double sequence $\langle V_n, W_n : n < \omega \rangle$ of open sets of X is *special* if each one of the U_n's occurs among the V_n's and among the W_n's and if for all $m, n < \omega$, $V_n \cap V_m = \varnothing$ iff $W_n \cap W_m = \varnothing$. For any special double sequence $\langle V_n, W_n : < \omega \rangle$ we define a relation G on X as follows:

$$xGy \text{ iff, for every } n < \omega,\ x \in V_n \Leftrightarrow y \in W_n .$$

We shall now prove that G is a one-one function by showing that for all $x_1, x_2, y_1, y_2 \in X$, if $x_1 G y_1$ and $x_2 G y_2$ then $x_1 = x_2$ iff $y_1 = y_2$. Assume $x_1 G y_1$, $x_2 G y_2$. If $x_1 \neq x_2$ then, since X is Hausdorff and the sequence of the V_n's contains all U_n's, there are n, m such that $x_1 \in V_n$, $x_2 \in V_m$ and $V_n \cap V_m = \varnothing$. By the definition of G we have $y_1 \in W_n$, $y_2 \in W_m$. Since $V_n \cap V_m = \varnothing$ and the double sequence is special we have $W_n \cap W_m = \varnothing$, therefore $y_1 \neq y_2$. Similarly, if $y_1 \neq y_2$ then $x_1 \neq x_2$. Thus G is a one-one function on a subset of X; we shall now see that for the particular special double sequence defined above

$G \supseteq g$. Since g is on A it suffices to prove $G(x) = g(x)$ for each $x \in A$. We saw that, for all n, g maps $A \cap V_n$ onto $A \cap W_n$ hence, since g is a permutation of A, $x \in V_n \Leftrightarrow g(x) \in W_n$ and therefore $xGg(x)$, i.e., $G(x) = g(x)$. By the second countability axiom the number of the open sets of X is at most $2^{\aleph_0}$, hence the number of the special double sequences in X is at most $2^{\aleph_0}$, and the number of the functions G obtained from them as above is at most $2^{\aleph_0}$. We saw that every g as in the statement of the lemma is included in one of those G's.

Proof of Theorem 1 — Second Stage. Our present aim is to prevent A from having nontrivial autohomeomorphisms, or at least to have as few as possible such homeomorphisms. We define now a task Q_β, for $\beta < 2^{\aleph_0}$, which tries to prevent the function g_β of Lemma 2 from being an autohomeomorphism of A. Q_β is carried out (at step α) as follows. If we have

$$(*) \qquad \text{there is an } x \in X - B_\alpha - \mathrm{Dom}(g_\beta)$$

then let x_0 be some such x. If $x_0 \in A_\alpha$ we do nothing, if $x_0 \notin A_\alpha \cup B_\alpha$ we add to $A_\alpha \cup B_\alpha$ the set of type (iv) which contains x_0, putting x_0 in $A_{\alpha+1}$. In either case

$$x_0 \in A_{\alpha+1} - \mathrm{Dom}(g_\beta) \subseteq A - \mathrm{Dom}(g_\beta),$$

hence g_β (or, rather, $g_\beta \restriction A$) is not an autohomeomorphism of A. If $(*)$ does not hold, then if

$$(**) \qquad \text{there is an } x \in X - B_\alpha \text{ such that } g_\beta(x) \notin A_\alpha \cup \{f^{2n}(x) : n \in \mathbf{Z}\}$$

then let x_0 be some such x. If $x_0 \in A_\alpha$ and $g_\beta(x_0) \in B_\alpha$ we do nothing. If $x_0 \in A_\alpha$ and $g_\beta(x) \notin A_\alpha \cup B_\alpha$ then we add to $A_\alpha \cup B_\alpha$ the set of type (iv) which contains $g_\beta(x_0)$ putting $g_\beta(x_0)$ in $B_{\alpha+1}$. If $x_0 \notin A_\alpha \cup B_\alpha$ and $g_\beta(x_0) \in B_\alpha$ we add to $A_\alpha \cup B_\alpha$ the set of type (iv) which contains x_0, putting x_0 in $A_{\alpha+1}$. If $x_0 \notin A_\alpha \cup B_\alpha$ and $g_\beta(x_0) \notin A_\alpha \cup B_\alpha \cup \{f^{2n}(x_0) : n \in \mathbf{Z}\}$ then we add to $A_\alpha \cup B_\alpha$ the sets of type (iv) which contain x_0 and $g_\beta(x_0)$ putting x_0 in $A_{\alpha+1}$ and $g_\beta(x_0)$ in $B_{\alpha+1}$. If the same set of type (iv) contains both x_0 and $g_\beta(x_0)$ then, since $g_\beta(x_0) \notin \{f^{2n}(x_0) : n \in \mathbf{Z}\}$, $g_\beta(x_0) = f^{2n+1}(x_0)$ for some $n \in \mathbf{Z}$ and when we put x_0 in $A_{\alpha+1}$, $g_\beta(x_0)$ is automatically put in $B_{\alpha+1}$. Thus in all cases of $(**)$ we get $x_0 \in A_{\alpha+1} \subseteq A$ and $g_\beta(x_0) \in B_{\alpha+1} \subseteq B$, hence $g_\beta(x_0) \notin A$ and g_β is not an autohomeomorphism of A. We had to assume in $(**)$ that $g_\beta(x_0) \neq f^{2n}(x_0)$, since if $g_\beta(x_0) = f^{2n}(x_0)$ then, if $x_0 \in A_{\alpha+1}$, also, by (e), $g_\beta(x_0) = f^{2n}(x_0) \in A_{\alpha+1}$ and we cannot put $g_\beta(x_0)$ in $B_{\alpha+1}$ as we did in all the cases of $(**)$. If neither $(*)$ nor $(**)$ holds we do nothing.

Assume now that we have constructed the sequence $\langle A_\alpha, B_\alpha : \alpha < 2^{\aleph_0} \rangle$ and have carried out, at the recursion steps, various tasks, which include all the tasks P_β and Q_β, $\beta < 2^{\aleph_0}$. Let g be an autohomeomorphism of A. By Lemma 2 there are $\beta, \lambda < 2^{\aleph_0}$ such that $g \subseteq g_\beta$ and $g^{-1} \subseteq g_\lambda$. Suppose task Q_β was carried out when we constructed $A_{\alpha+1}$, $B_{\alpha+1}$ from A_α, B_α. If either (*) or (**) held then, as we saw above, g_β would not include an autohomeomorphism g of A, hence both (*) and (**) fail and we have, for every $x \in X - B_\alpha$, $x \in \mathrm{Dom}(g_\beta)$ and $g_\beta(x) \in A_\alpha \cup \{f^{2n}(x) : n \in \mathbf{Z}\}$. Since $A = X - B \subseteq X - B_\alpha$ and $g \subseteq g_\beta$ we have, for every $x \in A$, $g(x) \in A_\alpha \cup \{f^{2n}(x) : n \in \mathbf{Z}\}$. Suppose that the task Q_λ was carried out when we constructed $A_{\delta+1}$, $B_{\delta+1}$ from A_δ, B_δ then we have, similarly, that for every $x \in A$, $g^{-1}(x) \in A_\delta \cup \{f^{2n}(x) : n \in \mathbf{Z}\}$. Let $\gamma = \max(\alpha, \delta)$, then $A_\alpha, A_\delta \subseteq A_\gamma$ and we have

$$(\#) \qquad \text{for all } x \in A \qquad g(x), g^{-1}(x) \in A_\gamma \cup \{f^{2n}(x) : n \in \mathbf{Z}\}.$$

If $x \in A_\gamma$ then by (e) $\{f^{2n}(x) : n \in Z\} \subseteq A_\gamma$, hence the right-hand side of (#) equals A_γ and by (#) $g^{-1}(x) \in A_\gamma$. If $x \in A - A_\gamma$ and $g(x) \in A_\gamma$ then, by what we have just seen, $x = g^{-1}(g(x)) \in A_\gamma$, contradicting $x \in A - A_\gamma$. Therefore, by (#), for $x \in A - A_\gamma$ $g(x) \in \{f^{2n}(x) : n \in Z\}$, i.e., $g(x) = f^{2n(x)}(x)$. Thus we have established that if we carry out the appropriate tasks then:

(h) for every autohomeomorphism g of A there is a $\gamma < 2^{\aleph_0}$ and a function n on $A - A_\gamma$ into $\mathbf{Z}$ such that for every $x \in A - A_\gamma$, $g(x) = f^{2n(x)}(x)$.

Proof of Theorem 1 — Part A. By (h) we have a good hold on g on $A - A_\gamma$. Therefore we want to get:

(i) for every $\gamma < 2^{\aleph_0}$, $A - A_\gamma$ is dense in X.

This will also verify the assumption, which we have already used above, that A is dense in X. To obtain (i) we carry out, at the recursion steps, also the $2^{\aleph_0}$ following tasks $R_{\beta,n}$, $\beta < 2^{\aleph_0}$, $n < \omega$. $R_{\beta,n}$ adds to $A_\alpha \cup B_\alpha$ a set of type (iv) which contains x, putting x in $A_{\alpha+1}$, where x is some point in $U_n - A_\alpha \cup B_\alpha$, and U_n is the n-th basic open set. There is such an x by (5) and (f).

Let g be a nontrivial autohomeomorphism of A. Then $\{x \in A : g(x) \neq x\}$ is a nonvoid open set in A. Let U be an open set such that $U \cap A = \{x \in A : g(x) \neq x\}$; clearly $U \neq \emptyset$. Let V be an open subset of U as in (6). Since $A - A_\gamma$ is dense in X there is an $x_0 \in (A - A_\gamma) \cap V$, then $x_0 \in A \cap U$, hence $g(x_0) \neq x_0$. By (h), $g(x_0) = f^{2n(x_0)}(x_0)$; let us denote $n(x_0)$ with n. $n \neq 0$ since $g(x_0) \neq x_0$. By (6) $d(f^{2n}(V), \bigcup_{m \in \mathbf{Z}, m \neq 2n} f^m(V)) > 0$, therefore there is an open set $W^* \supseteq f^{2n}(V)$ such that $f^m(V) \cap W^* = \emptyset$ for $m \neq 2n$. We have $x_0 \in V$, $g(x_0) = f^{2n}(x_0) \in f^{2n}(V) \subseteq W^*$. Since $g(x_0) \neq x_0$ there are disjoint open sets W_0, W_1 such that $x_0 \in W_0$ and $g(x_0) \in W_1$. Without loss of generality $W_1 \subseteq W^*$ and $W_0 \subseteq V$. Since g is continuous there is an open set W_2 such that $x_0 \in W_2 \subseteq W_0$ and $g(W_2 \cap A) \subseteq W_1 \cap A$. For every $x \in W_2 \cap (A - A_\gamma)$,

$g(x) \in W_1 \subseteq W^*$ and $g(x) = f^{2n(x)}(x)$, hence, by our choice of W^*, $n(x) = n$. Thus $g(x) = f^{2n}(x)$ for every $x \in W_2 \cap (A - A_\gamma)$. The functions $f^{2n} \restriction (W_2 \cap \operatorname{Dom} f^{2n} \cap A)$ and $g \restriction (W_2 \cap \operatorname{Dom} f^{2n} \cap A)$ are continuous functions on $W_2 \cap \operatorname{Dom} f^{2n} \cap A$ which coincide on the set $W_2 \cap (A - A_\gamma)$ which is, by (i), dense in $W_2 \cap \operatorname{Dom} f^{2n} \cap A$. Therefore

$$g \restriction (W_2 \cap \operatorname{Dom} f^{2n} \cap A) = f^{2n} \restriction (W_2 \cap \operatorname{Dom} f^{2n} \cap A).$$

By (7) for W_2 and n there are $x_i \in W_2 \cap D_1 \cap \operatorname{Dom} f^{2n}$, $i < \omega$, such that the sequence $\langle x_i : i < \omega \rangle$ converges to some point $x_\omega \in D_1$ but $\langle f^{2n}(x_i) : i < \omega \rangle$ does not converge to any point in $X - D_2$. Since

$$x_i \in W_2 \cap D_1 \cap \operatorname{Dom} f^{2n} \subseteq W_2 \cap A \cap \operatorname{Dom} f^{2n},$$

we have, by what we have just now shown, $g(x_i) = f^{2n}(x_i)$ for $i < \omega$, hence

$$\langle g(x_i) : i < \omega \rangle = \langle f^{2n}(x_i) : i < \omega \rangle.$$

Since $x_\omega \in D_1 \subseteq A$ and g is continuous

$$\lim_{i \to \infty} f^{2n}(x_i) = \lim_{i \to \infty} g(x_i) = g(x_\omega).$$

$g(x_\omega) \in A \subseteq X - D_2$, contradicting what we said about $\langle f^{2n}(x_i) : i < \omega \rangle$.

Proof of Theorem 1 — Part B. Let $\langle U_\beta, C_\beta : \beta < 2^{\aleph_0} \rangle$ enumerate all pairs $\langle U, C \rangle$ where U is a nonvoid open set and C is a countable union of closed nowhere dense subsets of X, each pair appearing $2^{\aleph_0}$ times. In the construction of $\langle A_\alpha, B_\alpha : \alpha < 2^{\aleph_0} \rangle$ we replace the tasks $R_{\beta,n}$ by the tasks R_β, $\beta < 2^{\aleph_0}$, where the task R_β is to take an $x \in U_\beta - (A_\alpha \cup B_\alpha) - C_\beta$ and put in $A_{\alpha+1} \cup B_{\alpha+1}$ the set of type (iv) which contains x, putting x in $A_{\alpha+1}$. $U_\beta - (A_\alpha \cup B_\alpha) - C_\beta \neq \emptyset$ since, by $(6)^B$ and (f), $U_\beta - (A_\alpha \cup B_\alpha)$ is not meager. Having carried out the tasks $R_\beta, \beta < 2^{\aleph_0}$, we get that for every $\gamma < 2^{\aleph_0}$ and every nonvoid open set U, $U \cap (A - A_\gamma)$ has a point outside each meager set. Thus

(j) for every $\gamma < 2^{\aleph_0}$ and every nonvoid open set U, $U \cap (A \cap A_\gamma)$ is not meager.

In particular we know that A is dense in X.

Let g be an autohomeomorphism of A which is not the identity. Since $\{x \in A : g(x) = x\}$ is a closed subset of A there is a nonvoid open set W such that $W \cap A \subseteq \{x \in A : g(x) \neq x\}$. Let $\gamma < 2^{\aleph_0}$ and the function n be as in (h), then for $x \in W \cap (A - A_\gamma)$, $n(x) \neq 0$ and

$$W \cap (A - A_\gamma) = \bigcup_{n \in \mathbf{Z},\, n \neq 0} \{x \in W \cap (A - A_\gamma) : n(x) = n\}.$$

Since $W \cap (A - A_\gamma)$ is not meager, one of the sets $\{x \in W \cap (A - A_\gamma) : n(x) = n\}$ is not nowhere dense; hence there is a nonvoid open set U such that for a dense subset V of U, $f^{2n} \restriction V = g \restriction V$. As in the proof of Part A we conclude that

$$g \restriction (U \cap \operatorname{Dom} f^{2n} \cap A) = f^{2n} \restriction (U \cap \operatorname{Dom} f^{2n} \cap A).$$

Using (7) as in the proof of Part A we get a contradiction.

PROOF OF THEOREM 1 — PART C. This is left to the reader.

PROOF OF THE MAIN THEOREM — FIRST STAGE

Proof of (3). Let U be an open nonvoid subset of X. Since X is separable, X is the union of $\aleph_0$ translations of U. Therefore, since $|X| = 2^{\aleph_0}$ also $|U|$ must be $2^{\aleph_0}$.

Let f be an autohomeomorphism of X of order 2 without fixed points. Take $D_1 = D_2 = \emptyset$. By the remark which we have just made, all the requirements of Theorem 1(C) are fulfilled and X can be partitioned to two homeomorphic rigid sets.

Proof of (1). We shall now prove the theorem under the assumption that (1) holds but X is not meager, since the case where X is meager is dealt with by (2). We shall prove later (Lemma 4) that the hypothesis of Theorem 1(B) holds for the case where X is the real line $\mathfrak{R}$ together with the additional demand that $D_1 \cup D_2$ is meager. Building on this we shall prove case (1) of the Main Theorem, and first we shall prove the following lemma.

LEMMA 3. *Let X be a nonmeager normed vector space over* $\mathbf{Q}$ *of cardinality* $2^{\aleph_0}$.

(a) *X is not the union of a meager set and a set of cardinality* $< 2^{\aleph_0}$.

(b) *No nonvoid open set U in X is the union of a meager set and a set of cardinality* $< 2^{\aleph_0}$.

PROOF OF LEMMA 3. (a) Let M be a meager set and $W \subseteq X$, $|W| < 2^{\aleph_0}$. Let $W^- = \{y - y' : y, y' \in W\}$; clearly $|W^-| < 2^{\aleph_0}$. Since $|X| = 2^{\aleph_0}$ there is a $z \in X - W^-$. By our choice of z, $W + z = \{y + z : y \in W\}$ is disjoint from W, hence $W + z \subseteq X - W$. If $M \cup W = X$ then $W + z \subseteq X - W \subseteq M$, hence

$W + z$ is meager and also its translation W is meager. Since $X = M \cup W$ also X is meager contradicting our hypothesis.

(b) Without loss of generality $0 \in U$. Assume that $U = \bigcup_{n\in\omega} M_n \cup W$, where each M_n is nowhere dense, and $|W| < 2^{\aleph_0}$. Then $X = \bigcup_{k,n\in\omega} kM_n \cup \bigcup_{k\in\omega} kW$ where, for $Y \subseteq X$, $kY = \{ky \mid y \in Y\}$. Since each M_n is nowhere dense $\bigcup_{k,n\in\omega} kM_n$ is meager. We have $|kW| < 2^{\aleph_0}$ and since a set of cardinality $2^{\aleph_0}$ is not the union of $\aleph_0$ sets of smaller cardinality, $|\bigcup_{k\in\omega} kW| < 2^{\aleph_0}$. Thus X is the union of a meager set and a set of cardinality $< 2^{\aleph_0}$, contradicting (a).

Proof of the Main Theorem — First Stage (continued). Let f^*, D_1^*, D_2^*, E_1^*, E_2^* be the respective function and sets which satisfy the hypothesis of Theorem 1(B) for the real line $\mathfrak{R}$ such that $D_1^* \cup D_2^*$ is meager in $\mathfrak{R}$. Let X be a separable normed vector space over $\mathbf{Q}$ and let $\mathfrak{R}z$, $z \neq 0$, be a complete direction in X. By the Hahn–Banach theorem there is a bounded linear functional P on X such that $P(z) = 1$. Let $D_i = P^{-1}D_i^*$, $E_i = P^{-1}E_i^*$ for $i = 1, 2$ and let, for $x \in X - E_2$,

$$f(x) = x + (f^*(P(x)) - P(x))z.$$

Aided by the observations that for all $x \in X - E_2$, $P(f(x)) = f^*(P(x))$, and that for $x \in X - E_1$, $f^{-1}(x) = x + (f^{*-1}(Px) - Px)z$, we can easily show that D_1, D_2, E_1, E_2 and f satisfy conditions (1)–(4) of Theorem 1.

We shall now prove that $(6)^B$ holds too. Let S be nowhere dense in $\mathfrak{R}$ then we shall see that $P^{-1}(S)$ is nowhere dense in X. Let U be a nonvoid open set in X and let $x \in U$, then for some $r > 0$, $\{x + tz : |t| < r\} \subseteq U$, hence the open interval $(P(x) - r, P(x) + r)$ is a subset of $P(U)$. Since S is nowhere dense $(P(x) - r, P(x) + r)$ has a nonvoid open subset V such that $V \cap S = \varnothing$. We have $P^{-1}(V) \cap P^{-1}(S) = \varnothing$. Since P is continuous, $P^{-1}(V)$ is open. Let $P(x) + t^* \in V$, $|t^*| < r$, then $x + t^*z \in P^{-1}(V) \cap U$ and thus $P^{-1}(V) \cap U$ is a nonvoid subset of U disjoint from $P^{-1}(S)$. As a consequence of what we have just proved, since $D_1^* \cup D_2^*$ is meager in $\mathfrak{R}$, $D_1 \cup D_2 = P^{-1}(D_1^* \cup D_2^*)$ is meager in X. For every nonvoid open set U in X, $|U - (D_1 \cup D_2)| = 2^{\aleph_0}$ by Lemma 3 (b), thus also condition $(6)^B$ holds.

We shall now see that condition (7) of Theorem 1 holds too. Let U be a nonvoid open set in X and let $x \in U$. Then there is an $r > 0$ such that $\{x + tz : |t| < r\} \subseteq U$. Since (7) holds for the open interval $(P(x) - r, P(x) + r)$ in $\mathfrak{R}$, it lifts easily, by P^{-1}, to hold for the open set U in X.

Since X satisfies requirements (1)–(4), $(6)^B$ and (7) then, by Theorem 1(B), X can be partitioned into two rigid homeomorphic sets.

Now we shall make the preparations needed in order to continue proving the Main Theorem.

NOTATION. Let $X = \langle X, d \rangle$ be a metric space and let $\mathfrak{R}^+$ be the set of all positive reals. For $x \in X$, $r \in \mathfrak{R}^+$ let the *open ball* $B(x, r)$ be $\{z \in X : d(x, z) < r\}$. For a bounded set V in X, $\operatorname{diam}(V)$ denotes the diameter of V. For any set U, $\operatorname{Bd}(U)$ is the boundary of U.

LEMMA 4. *Assume that*:

(1) $X = \langle X, d \rangle$ *is a complete separable metric space without isolated points.*

(2) F *is an autohomeomorphism of* X *such that*

(a) *for every* $x \in X$ *and* $n > 0$, $F^{2n+1}(x) \neq x$, *and*

(b) *every nonvoid open set* U *has a nonvoid open subset* V *such that for each* $n \in \mathbf{Z}$, $d(F^n(V), \bigcup_{m \in \mathbf{Z}, m \neq n} F^m(V)) > 0$.

(3) *Every nonvoid open set* U *has two nonvoid disjoint open subsets* V *and* W *and a homeomorphism* g *from* V *onto* W *such that for every closed set* C, *if* $C \subseteq V$ *then also* $g(C)$ *is closed, and if* $C \subseteq W$ *then also* $g^{-1}(C)$ *is closed.*

Then *there are* f, D_1, D_2, E_1, E_2 *which satisfy the hypotheses of Theorem* 1(B) *and such that* $D_1 \cup D_2$ *is meager.*

We shall first give a definition and prove a lemma (lemma 6) and then we shall prove Lemma 4.

DEFINITION 5. An *approximation* T consists of:

(α) Four sets D_1^T, D_2^T, E_1^T, E_2^T such that $E_1^T \subseteq D_1^T$, $E_2^T \subseteq D_2^T$, E_1^T and E_2^T are closed, and D_1^T and D_2^T are nowhere dense disjoint sets.

(β) A homeomorphism f_T from $X - E_2^T$ onto $X - E_1^T$.

f_T, D_1^T, E_1^T, D_2^T, E_2^T are such that the following (γ)–(ε) hold:

(γ) f_T maps D_1^T onto D_2^T and $D_2^T - E_2^T$ onto $D_1^T - E_1^T$, hence $D_1^T \cup D_2^T$ is closed under f_T and f_T^{-1}.

(δ) Every nonvoid open set U has a nonvoid open subset $V \subseteq X - (D_1^T \cup D_2^T)$ such that the sets $f_T^n(V)$, $n \in \mathbf{Z}$, are open, for every $n \in \mathbf{Z}$, $d(f_T^n(V), \bigcup_{m \in \mathbf{Z}, m \neq n} f_T^m(V)) > 0$, and for all $m, n \in \mathbf{Z}$ and for every closed set $C \subseteq f_T^m(V)$, $f_T^n(C)$ is closed too.

(ε) For every $x \in X$ and $n \geqq 0$, if f_T^{2n+1} is defined then $f_T^{2n+1}(x) \neq x$.

Such a T is called an approximation because D_1^T, D_2^T, E_1^T, E_2^T, f_T are an approximation to D_1, D_2, E_1, E_2, f in the conclusion of Lemma 4.

LEMMA 6. *For every approximation S, $\varepsilon > 0$, $k > 0$ and nonvoid open set U there is an approximation T such that*:

(1) *There is a sequence $\langle z_i : i < \omega \rangle$ of points of $D_1^T \cap U$ which converges to a point of $D_1^T \cap U$ and for every $m \neq 0$ the sequence $\langle f_T^{2m}(z_i) : i < \omega \rangle$ is defined but does not converge.*

(2) $E_l^S \subseteq E_l^T$, $D_l^S \subseteq D_l^T$ *and* $(E_l^T - E_l^S) \cap D_l^S = \varnothing$ *for* $l = 1, 2$; *hence* $\mathrm{Dom}\, f_T \subseteq \mathrm{Dom}\, f_S$, $\mathrm{Dom}\, f_T^{-1} \subseteq \mathrm{Dom}\, f_S^{-1}$.

(3) f_T *agrees with* f_S *on* $D_1^S \cup D_2^S$.

(4) *For every* $x \in \mathrm{Dom}\, f_T$, $d(f_T(x), f_S(x)) \leqq \varepsilon \cdot \min(1, d(f_S(x), E_1^S))$.

(5) *For every* $x \in \mathrm{Dom}\, f_T^{-1}$,

$$d(f_T^{-1}(x), f_S^{-1}(x)) \leqq \varepsilon \cdot \min(1, d(f_T^{-1}(x), f_S^{-1}(x)))$$

$$\leqq \varepsilon \cdot \min(1, d(f_S^{-1}(x), E_2^S)).$$

(6) *For every* $1 \leqq i \leqq k$ *and* $x \in \mathrm{Dom}\, f_T^i$, $d(f_T^i(x), x) \geqq (1 - \varepsilon) d(f_S^i(x), x)$.

(7) $D_2^T \cap U \neq \varnothing$

PROOF OF LEMMA 6. Let S, ε, k, U be given. By Definition 5(δ), U has a nonvoid open subset $V \subseteq X - (D_1^S \cup D_2^S)$ such that the sets $V_m = f_S^m(V)$, $m \in \mathbf{Z}$, are open, and for every $m \in \mathbf{Z}$, $d(V_m, \bigcup_{i \in \mathbf{Z}, i \neq m} V_i) > 0$, and for every closed set $C \subseteq V_m$ also $f_S(C)$ is closed. We shall now shrink V so that also the following conditions (i)–(v) will be satisfied:

(i) $d(V, E_2^S) > 0$.

(ii) $d(V_1, E_1^S) > 0$.

(iii) For $i = -1, 1$, $\mathrm{diam}(V_i) \leqq \varepsilon \cdot \min(1, d(V_i, E_2^S))$.

(iv) For $i = 0, 2$, $\mathrm{diam}(V_i) \leqq \varepsilon \cdot \min(1, d(V_i, E_1^S))$.

(v) $\mathrm{diam}(V)$, $\mathrm{diam}(V_1) \leqq \varepsilon \cdot \min_{-k \leq j \leq 0} d(V_j, \bigcup_{m \in \mathbf{Z}, m \neq j} V_m)$.

Let us notice that if we replace V by a nonvoid open subset V' of V, what we have assumed above about V holds also for V' and those requirements among (i)–(v) which are satisfied by V are also satisfied by V'; therefore we can deal with each one of (i)–(v) separately. First let us check one point. f_S is a homeomorphism of $X - E_2^S$ onto $X - E_1^S$, by Definition 5(γ) $D_1^S \cup D_2^S$ is closed under f_S, and $V \subseteq X - (D_1^S \cup D_2^S)$; hence, for $m \in \mathbf{Z}$, f_S^m is a homeomorphism of V onto $f_S^m(V)$. Since V' is open $f_S^m(V')$ is open in the relative topology of $f_S^m(V)$, and since $f_S^m(V)$ is open so is also $f_S^m(V')$.

Now let us deal with (i)–(ii). In each one of those cases we have to shrink V so that two sets which we know already to be disjoint will have positive distance between them. For (i) this is immediate and for (ii) this follows from the continuity of f_S. Now we shrink V further to meet requirements (iii)–(v): also this is possible since each one of the functions f_S, f_S^2, f_S^{-1} is continuous.

By Lemma 4(3) V has nonvoid disjoint open subsets M, N and a homeomorphism g from M onto N such that for every closed set C, if $C \subseteq M$ then also $g(C)$ is closed, and if $C \subseteq N$ then also $g^{-1}(C)$ is closed. Choose $\bar{x} \in M$ and let $c = d(\bar{x}, X - M)$, $\bar{y} = g(\bar{x})$. Since X has no isolated points there is a sequence $c > r_{-1} > r_0 > r_1 > \cdots$ of positive reals and a sequence $z_{-1}, z_0, z_1, \ldots$ of points of X such that $\lim_{i \to \infty} r_i = 0$ and $d(z_i, \bar{x}) = r_i$, for $i < \omega$. For $i, l < \omega$ let $s_i = \frac{1}{2}(r_{i-1} + r_i)$, $M_i = B(\bar{x}, s_i)$, $N_i = g(M_i)$, $M_{l,i} = f_S^l(M_i)$, $N_{l,i} = f_S^l(N_i)$, $P_i = \mathrm{Bd}(M_i)$, $Q_i = g(P_i)$, $P_{l,i} = f_S^l(P_i)$, $Q_{l,i} = f_S^l(Q_i)$. We now define T as follows:

$$E_1^T = E_1^S \cup \bigcup_{i<\omega} P_i \cup \bigcup_{i<\omega} Q_i \cup \{\bar{x}, \bar{y}\}$$

$$\cup \bigcup_{i<\omega} P_{2,i} \cup \bigcup_{i<\omega} Q_{2,i} \cup \{f_S^2(\bar{x}), f_S^2(\bar{y})\},$$

$$E_2^T = E_2^S \cup \bigcup_{i<\omega} P_{1,i} \cup \bigcup_{i<\omega} Q_{1,i} \cup \{f_S(\bar{x}), f_S(\bar{y})\}$$

$$\cup \bigcup_{i<\omega} P_{-1,i} \cup \bigcup_{i<\omega} Q_{-1,i} \cup \{f_S^{-1}(\bar{x}), f_S^{-1}(\bar{y})\}.$$

For $x \notin E_2^T$ we set

$$f_T(x) = \begin{cases} f_S^2(g(f_S^{-1}(x))) & \text{if } x \in \bigcup_{i<\omega} (M_{1,2i} - M_{1,2i+1} - P_{1,2i+1}), \\ f_S^2(g^{-1}(f_S^{-1}(x))) & \text{if } x \in \bigcup_{i<\omega} (N_{1,2i} - N_{1,2i+1} - Q_{1,2i+1}), \\ g(f_S(x)) & \text{if } x \in \bigcup_{i<\omega} (M_{-1,2i} - M_{-1,2i+1} - P_{-1,2i+1}), \\ g^{-1}(f_S(x)) & \text{if } x \in \bigcup_{i<\omega} (N_{-1,2i} - N_{-1,2i+1} - Q_{-1,2i+1}), \\ f_S(x) & \text{otherwise;} \end{cases}$$

$$D_1^T = D_1^S \cup \bigcup_{l \in Z} \left(\bigcup_{i<\omega} P_{2l,i} \cup \bigcup_{i<\omega} Q_{2l,i} \cup \{f_S^{2l}(\bar{x}), f_S^{2l}(\bar{y})\} \right.$$

$$\left. \cup \{f_T^{2l}(z_i) : i < \omega\} \cup \{f_S^{2l+1}(z_{-1})\} \right),$$

$$D_2^T = D_2^S \cup \bigcup_{l \in Z} \left(\bigcup_{i<\omega} P_{2l+1,i} \cup \bigcup_{i<\omega} Q_{2l+1,i} \cup \{f_S^{2l+1}(\bar{x}), f_S^{2l+1}(\bar{y})\} \right.$$

$$\left. \cup \{f_T^{2l+1}(z_i) : i < \omega\} \cup \{f_S^{2l}(z_{-1}\}\right).$$

Let us prove now that T is an approximation, i.e., it satisfies requirements (α)–(ε) of Definition 5.

(α, β) First we prove that E_1^T and E_2^T are closed nowhere dense sets. For $i < \omega$, P_i is the boundary of $B(\bar{x}, s_i)$, hence it is a closed nowhere dense set. It is also easily seen that $\bigcup_{i<\omega} P_i \cup \{\bar{x}\}$ is closed. For each $i < \omega$, $d(P_i, \bigcup_{j<\omega, j\neq i} P_j)$ is $s_0 - s_1 > 0$ for $i = 0$ and

$$\min(s_{i-1} - s_i, s_i - s_{i+1}) > 0 \qquad \text{for } i > 0.$$

As a consequence $\bigcup_{i<\omega} P_i$ is nowhere dense.

For a subset P of an open set W, P is obviously nowhere dense if it is nowhere dense in the relative topology of W, and we shall use this fact tacitly from now on. f_S is a homeomorphism of V on V_1 and V, V_1 are open, hence since $\bigcup_{i<\omega} P_i$ is nowhere dense, also $\bigcup_{i<\omega} P_{1,i} = f_S(\bigcup_{i<\omega} P_i)$ is nowhere dense. For the same reason also $\bigcup_{i<\omega} P_{j,i}$, $j = -1, 2$, $\bigcup_{i<\omega} Q_i$, and $\bigcup_{i<\omega} Q_{j,i}$, $i, j = -1, 1, 2$, are nowhere dense. E_1^S and E_2^S are nowhere dense, since S is an approximation, therefore E_1^T and E_2^T are nowhere dense, being finite unions of nowhere dense sets. By Definition 5(δ), f_S and f_S^{-1} map closed subsets of V onto closed sets hence

$$f_S\left(\bigcup_{i<\omega} P_i \cup \{\bar{x}\} \right) = \bigcup_{i<\omega} P_{1,i} \cup \{f_S(\bar{x})\}$$

is closed. Similarly also $\bigcup_{i<\omega} P_{j,i} \cup \{f_S^j(\bar{x})\}$, $j = -1, 2$, are closed. By our choice of g, g maps closed subsets of M onto closed subsets of N hence

$$g\left(\bigcup_{i<\omega} P_i \cup \{\bar{x}\} \right) = \bigcup_{i<\omega} Q_i \cup \{\bar{y}\}$$

is closed, and as we saw just now, also

$$f_S^j\left(\bigcup_{i<\omega} Q_i \cup \{\bar{y}\} \right) = \bigcup_{i<\omega} Q_{j,i} \cup \{f_S^j(\bar{y})\}, \qquad j = -1, 1, 2,$$

are closed. E_1^S and E_2^S are closed by Definition 5(α), hence also E_1^T and E_2^T are closed.

Our next aim is to prove that f_T is a homeomorphism of $X - E_2^T$ onto $X - E_1^T$. Looking at the definition of f_T we see that $\operatorname{Dom} f_T$ decomposes disjointly to the sets

$$(*)\quad \begin{cases} M_{1,2i} - M_{1,2i+1} - P_{1,2i+1}, \quad N_{1,2i} - N_{1,2i+1} - Q_{1,2i+1}, \\ M_{-1,2i} - M_{-1,2i+1} - P_{-1,2i+1}, \quad N_{-1,2i} - N_{-1,2i+1} - Q_{-1,2i+1} \quad \text{for } i < \omega \\ \text{and} \\ Y = \text{the complement in } X - E_2^T \text{ of the union of these sets.} \end{cases}$$

f_T, as defined, is a homeomorphism on each of these sets, being equal either to f_S or else to a composition of homeomorphisms from among f_S, f_S^{-1}, g, g^{-1}. The images of the sets (*) under f_T are, respectively, the pairwise disjoint sets

$$(**)\quad \begin{cases} N_{2,2i} - N_{2,2i+1} - Q_{2,2i+1}, \quad M_{2,2i} - M_{2,2i+1} - P_{2,2i+1}, \\ N_{2i} - N_{2i+1} - Q_{2i+1}, \quad M_{2i} - M_{2i+1} - P_{2i+1} \\ \text{and} \\ Z = \text{the complement in } X - E_1^T \text{ of the union of these sets.} \end{cases}$$

This is easily seen since f_S is a bijection of $X - E_2^S$ onto $X - E_1^S$, and $f_S(E_2^T - E_2^S) = E_1^T - E_1^S$. Therefore, in order to establish that f_T is a homeomorphism it suffices to show that each one of the sets of (*) and (**) is clopen in $X - E_2^T$ and $X - E_1^T$, respectively. The set $M_{2i} - M_{2i+1} - P_{2i+1}$ is open. The set $M_{2i} \cup P_{2i} - M_{2i+1}$ is closed. By our choice of g, $N_{2i} - N_{2i+1} - Q_{2i+1}$ is open and $N_{2i} \cup Q_{2i} - N_{2i+1}$ is closed. By Definition 5(β) and (δ) and our choice of U, also all the sets $M_{j,2i} - M_{j,2i+1} - P_{j,2i+1}$ are open and the sets $M_{j,2i} \cup P_{j,2i} - M_{j,2i+1}$ are closed. Clearly

$$\begin{aligned} &M_{j,2i} - M_{j,2i+1} - P_{j,2i+1} \\ &\quad = (M_{j,2i} \cup P_{j,2i} - M_{j,2i+1}) \cap (X - E_2^T) \qquad \text{for } j = -1, 1 \end{aligned}$$

hence the sets $M_{j,2i} - M_{j,2i+1} - P_{j,2i+1}$ are clopen in $X - E_2^T$ for $j = -1, 1$, and similarly also the sets $N_{j,2i} - N_{j,2i+1} - Q_{j,2i+1}$ are clopen in $X - E_2^T$. Similarly also the sets $M_{j,2i} - M_{j,2i+1} + P_{j,2i+1}$ and $N_{j,2i} - N_{j,2i+1} - Q_{j,2i+1}$, for $j = 0, 2$, are clopen in $X - E_1^T$. As easily seen, the closure of the union $\bigcup_{i<\omega}(M_{2i} \cup P_{2i} - M_{2i+1})$ of closed sets is $\bigcup_{i<\omega}(M_{2i} \cup P_{2i} - M_{2i+1}) \cup \{\bar{x}\}$ and hence also its images

$$\bigcup_{i<\omega} (M_{j,2i} \cup P_{j,2i} - M_{j,2i+1}) \cup \{f_S^j(\bar{x})\}, \qquad j = -1, 1, 2,$$

and $\bigcup_{i<\omega}(N_{j,2i} \cup P_{j,2i} - N_{j,2i+1}) \cup \{f_S^j(\bar{y})\}$, $j = -1, 0, 1, 2$, are closed. For $j = -1, 1$, $\bigcup_{i<\omega}(M_{j,2i} - M_{j,2i+1} - P_{j,2i+1})$ is open, being the union of open sets. Since

$$\bigcup_{i<\omega} (M_{j,2i} - M_{j,2i+1} - P_{j,2i+1})$$

$$= \bigcup_{i<\omega} (M_{j,2i} \cup P_{j,2i} - M_{j,2i+1}) \cup \{f_S^j(\bar{x})\}) \cap (X - E_2^T)$$

these sets are also closed in $X - E_2^T$, hence they are clopen in $X - E_2^T$. Similarly also the sets $\bigcup_{i<\omega}(N_{j,2i} - N_{j,2i+1} - Q_{j,2i+1})$ are clopen in $X - E_2^T$ and therefore also the set Y of (*) is clopen in $X - E_2^T$. Similarly also the set Z of (**) is clopen in $X - E_1^T$. This establishes our claim that f_T is a homeomorphism of $X - E_2^T$ onto $X - E_1^T$.

Now let us see that D_1^T and D_2^T are nowhere dense. By Definition 5(α), D_1^S is nowhere dense. We proved above that E_1^T is nowhere dense, thus its subset $\bigcup_{i<\omega} P_i \cup \bigcup_{i<\omega} Q_i \cup \{\bar{x}, \bar{y}\}$ is a nowhere dense subset of V. Therefore, also its image by f_S^{2l}, $\bigcup_{i<\omega} P_{2l,i} \cup \bigcup_{i<\omega} Q_{2l} \cup \{f_S^{2l}(\bar{x}), f_S^{2l}(\bar{y})\}$ is a nowhere dense subset of V_{2l}. Since each of the different V_i's has a positive distance from the union of the other ones also $\bigcup_{l\in\mathbf{Z}} (\bigcup_{i<\omega} P_{2l,i} \cup \bigcup_{i<\omega} Q_{2l,i} \cup \{f_S^{2l}(\bar{x}), f_S^{2l}(\bar{y})\}$ is nowhere dense. As easily seen from the definition of f_T, for every $m \neq 0, 1$

$$\{f_T^m(z_i) : i < \omega\} = f_S^m g\{z_{2i} : i < \omega\} \cup f_S^m\{z_{2i+1} : i < \omega\},$$

and for $m = 0, -1$

$$\{f_T^m(z_i) : i < \omega\} = f_S^m\{z_i : i < \omega\}.$$

Since, as can be easily seen, $\{z_i : i < \omega\}$ is nowhere dense we have, as above, that the set $\bigcup_{l\in\mathbf{Z}}\{f_T^{2l+1}(z_i) : i < \omega\}$ is nowhere dense. Similarly, also $\{f_S^{2l}(z_{-1}) : l \in Z\}$ is nowhere dense. Thus D_1^T is nowhere dense, being the union of finitely many nowhere dense sets. Similarly also D_2^T is nowhere dense.

(γ) To see the effect of applying f_T to D_1^T and D_2^T we shall see where f_T differs from f_S. By the definition of f_T it coincides with f_S outside

$$M_1 \cup N_1 \cup M_{-1} \cup N_{-1} \subseteq V_1 \cup V_{-1}.$$

By our choice of V, $V \cap (D_1^S \cup D_2^S) = \emptyset$, and since $D_1^S \cup D_2^S$ is closed under f_S also

$$V_1 \cap (D_1^S \cup D_2^S) = \emptyset \quad \text{and} \quad V_{-1} \cap (D_1^S \cup D_2^S) = \emptyset,$$

hence

$$(D_1^S \cup D_2^S) \cap (V_1 \cup V_{-1}) = \emptyset.$$

All the components of D_1^T and D_2^T given in their definitions, other than D_1^S and D_2^S, are included in some V_i, for $i \neq 1, -1$, and hence disjoint from $V_1 \cup V_{-2}$, except for $\{f_S(z_{-1})\}$, $\{f_S^{-1}(z_{-1})\}$, $P_{1,i}$, $Q_{1,i}$, $P_{-1,i}$, $Q_{-1,i}$, $\{f_S(\bar{x}), f_S(\bar{y})\}$, $(f_S^{-1}(\bar{x}), f_S^{-1}(\bar{y})\}$, $\{f_T(z_i) : i < \omega\}$, $\{f_T^{-1}(z_i) : i < \omega\}$, and $f_S(z_{-1}) \in V_1 - M_{1,0}$, $f_S^{-1}(z_{-1}) \in V_{-1} - M_{-2,0}$. Looking at the definition of D_1^T and D_2^T we see now easily that $f_T(D_1^T) = f_S(D_1^T) = D_2^T$, and since, as easily seen, $f_S(E_2^T - E_2^S) = E_1^T - E_1^S$, we get also $f_T(D_2^T - E_2^T) = D_1^T - E_1^T$.

(δ) Let $\bar{U}$ be a nonvoid open set, we have to prove that it has a nonvoid open subset $\bar{V}$ as required by (δ). Since S is an approximation, $\bar{U}$ has a nonvoid open subset V^* which satisfies (δ) for S. Since, as we have seen, $D_1^T \cup D_2^T$ is nowhere dense we can take $V^* \cap (D_1^T \cup D_2^T) = \emptyset$. We shall now distinguish several cases. *Case a*: V^* has a nonvoid open subset $\bar{V}$ such that $f_S^n(\bar{V}) \cap V = \emptyset$ for every $n \in \mathbf{Z}$. If *Case a* does not hold then, for some $n \in \mathbf{Z}$, $f_S^n(V^*) \cap V \neq \emptyset$.

$$(*) \qquad \begin{aligned} V = (V - M_0 - P_0 - N_0 - Q_0) \cup \bigcup_{i<\omega} P_i \cup \bigcup_{i<\omega} Q_i \cup \{\bar{x}\} \cup \{\bar{y}\} \\ \cup \bigcup_{i<\omega} (M_i - M_{i+1} - P_{i+1}) \cup \bigcup_{i<\omega} (N_i - N_{i+1} - Q_{i+1}) \end{aligned}$$

Since $f_S^n(V^*) \cap V$ is open and nonvoid and

$$\bigcup_{i<\omega} P_i \cup \bigcup_{i<\omega} Q_i \cup \{\bar{x}\} \cup \{\bar{y}\}$$

is nowhere dense, $f_S^n(V^*)$ must have a nonvoid intersection with one of the other sets which make up (*), i.e., with one of the open sets $V - M_0 - P_0 - N_0 - Q_0$, $M_i - M_{i+1} - P_{i+1}$, $N - N_{i+1} - Q_{i+1}$, where $i < \omega$. Let W be the nonvoid intersection of $f_S^n(V^*)$ with one of these sets, deleting from it one of the z_i's, for $i \in \{-1\} \cup \omega$, if this z_i is in it. W is obviously open. Let $\bar{V} = f_S^{-n}(W) \subseteq V^*$. Since f_S^{-n} is an autohomeomorphism of the open set $\bigcup_{i \in \mathbf{Z}} V_i$, $\bar{V}$ is open and $f_S^n(\bar{V}) = W$. We distinguish now also the following cases. *Case b*: $f_S^n(\bar{V}) \subseteq V - M_0 - P_0 - N_0 - Q_0$ or for some $i < \omega$, $f_S^n(\bar{V}) \subseteq M_{2i-1} - M_{2i} - P_{2i} - \{z_{2i-1}\}$ or $f_S^n(\bar{V}) \subseteq N_{2i-1} - N_{2i} - Q_{2i}$. *Case c*: For some

$i<\omega$, $f_S^n(\bar{V})\subseteq M_{2i}-M_{2i+1}-P_{2i+1}-\{z_{2i}\}$. *Case d*: For some $i<\omega$, $f_S^n(\bar{V})\subseteq N_{2i}-N_{2i+1}-Q_{2i+1}$.

Cases a, b: In these cases, by the definition of f_T, f_T agrees with f_S on every set $f_S^m(\bar{V})$. Hence for all $m\in\mathbf{Z}$, $f_T^m(\bar{V})=f_S^m(\bar{V})$, and ($\delta$) follows easily from the fact that V^* satisfies (δ) for S.

Case c: In this case $f_S^n(\bar{V})\subseteq M_{2i}-M_{2i+1}-P_{2i+1}-\{z_{2i}\}$ and hence

$$\bar{V}\subseteq M_{-n,2i}-M_{-n,2i+1}-P_{-n,2i+1}-\{f_S^{-n}(Z_{2i})\}.$$

Thus for every $m\in\mathbf{Z}$, $f_T^m(\bar{V})\subseteq M_{m-n,2i}\subseteq V_{m-n}$ or $f_T^m(\bar{V})\subseteq N_{m-n,2i}\subseteq V_{m-n}$ and hence $\bar{V}$ satisfies the first part of (δ). Powers of f_S preserve the property that a subset of V_m is closed, for $m\in\mathbf{Z}$, and so do g and g^{-1} for subsets of M and N, respectively. Since $f_T^m(\bar{V})\subseteq M$ or $f_T^m(\bar{V})\subseteq N$ and powers of f_T are compositions of powers of f_S and g also powers of f_T preserve the property of being closed for subsets of $f_T^k(\bar{V})$, for $k\in\mathbf{Z}$.

Case d: Similar to Case c.

(ε) If $x\notin\bigcup_{m\in\mathbf{Z}}V_i$, then if $f_T^{2n+1}(x)$ is defined then $f_S^{2n+1}(x)$ is defined and then $f_T^{2n+1}(x)=f_S^{2n+1}(x)$. Since S is an approximation, $f_T^{2n+1}(x)=f_S^{2n+1}(x)\neq x$. If $x\in V_m$ for some $m\in\mathbf{Z}$, then clearly $f_T^{2n+1}(x)\in V_{m+2n+1}$, and since $V_{m+2n+1}\cap V_m=\varnothing$, $f_T^{2n+1}(x)\neq x$.

Having shown that T is an approximation, we have to show that it satisfies requirements (1)–(7) of Lemma 6.

(1) The sequence $\langle z_i : i<\omega\rangle$ converges in $D_1^T\cap U$ to $\bar{x}\in D_1^T\cap U$. For $n\neq 0$

$$f_T^{2n}(z_i)=\begin{cases} f_S^{2n}g(z_i) & \text{if } i \text{ is even,}\\ f_S^{2n}(z_i) & \text{if } i \text{ is odd.}\end{cases}$$

Since $\lim_{i\to\infty} z_i=\bar{x}$ we have

$$\lim_{i\to\infty,\, i \text{ is even}} f_T^{2n}(z_i)=\lim_{i\to\infty} f_S^{2n}g(z_i)=f_S^{2n}g(\bar{x})=f_S^{2n}(\bar{y}),$$

$$\lim_{i\to\infty,\, i \text{ is odd}} f_T^{2n}(z_i)=\lim_{i\to\infty} f_S^{2n}(z_i)=f_S^{2n}(\bar{x}).$$

Since $\bar{x}\in M$, $\bar{y}\in N$ and $M\cap N=\varnothing$, $\bar{x}\neq\bar{y}$ and $f_S^{2n}(\bar{x})\neq f_S^{2n}(\bar{y})$, thus the sequence $\langle f_T^{2n}(z_i) : i<\omega\rangle$ does not converge.

(2) and (3) follow immediately from the definition of T.

(4) If $x\notin V_{-1}\cup V_1$ then $f_T(x)=f_S(x)$, hence $d(f_T(x),f_S(x))=0$. If $x\in V_i$, $i=-1,1$, then $f_T(x), f_S(x)\in V_{i+1}$, and hence $d(f_T(x),f_S(x))\leqq \operatorname{diam}(V_{i+1})$, $d(f_S(x),E_1^S)\geqq d(V_{i+1},E_1^S)$. By (iv), at the beginning of this proof,

$$\operatorname{diam}(V_{i+1}) \leqq \varepsilon \cdot \min(1, d(f_S(x), E_1^S))$$

hence $d(f_T(x), f_S(x)) \leqq \operatorname{diam}(V_{i+1}) \leqq \varepsilon \cdot \min(1, d(f_S(x), E_1^S)$.

(5) This is similar to the proof of (4), using the inequality (iii) instead of (iv).

(6) Let $1 \leqq i \leqq k$. If $f_T^i(x) = f_S^i(x)$ then (6) holds trivially. This is always the case unless $x \in V_{-i}$ or $x \in V_{-i+1}$ in which case $f_S^i(x), f_T^i(x)$ are in V_l where $l \in \{0, 1\}$.

$$d(f_T^i(x), x) \geqq d(f_S^i(x), x) - d(f_S^i(x), f_T^i(x)) \geqq d(f_S^i(x), x) - \operatorname{diam}(V_l),$$

and by the inequality (v) at the beginning of the proof

$$\geqq d(f_S^i(x), x) - \varepsilon \cdot \min_{-k<j\leqq 0} d\left(V_j, \bigcup_{m\in\mathbf{Z},\, m\neq j} V_m\right)$$

$$\geqq d(f_S^i(x), x) - \varepsilon \cdot d(V_{-i}, V_l)$$

$$\geqq d(f_S^i(x), x) - \varepsilon \cdot d(f_S^i(x), x) = (1-\varepsilon) d(f_S^i(x), x).$$

(7) $z_{-1} \in D_2^T \cap U$.

This completes the proof of Lemma 6.

Proof of Lemma 4. Let $\{U_n : n < \omega\}$ be a basis of the topology of X. We define a sequence of approximations $\langle T(n) : n < \omega \rangle$ such that for every n, $T(n+1)$ is related to $T(n)$ as T is related to S in Lemma 6, with $U = U_n$, $k = n$ and $\varepsilon = 2^{-(n+2)}$. $T(0)$ is defined by $D_1^{T(0)} = D_2^{T(0)} = E_1^{T(0)} = E_2^{T(0)} = \varnothing$, $f_{T(0)} = F$. It is easily seen that $T(0)$ is an approximation. Now we define $D_l = \bigcup_{n<\omega} D_l^{T(n)}$, $E_l = \bigcup_{n<\omega} E_l^{T(n)}$ for $l = 1, 2$. For $x \notin E_2$ we set $f(x) = \lim_{n\to\infty} f_{T(n)}(x)$. This limit exists since, by Lemma 6(4),

$$d(f_{T(n+1)}(x), f_{T(n)}(x)) \leqq 2^{-(n+2)},$$

hence the sequence $\langle f_{T(n)} : n < \omega \rangle$ is a uniformly convergent Cauchy sequence, and it has a limit f. Let us check now that conditions (1)–(5), $(6)^{\mathrm{B}}$ and (7) of Theorem 1(B) are satisfied.

(1) That $E_1 \subseteq D_1$, $E_2 \subseteq D_2$ and $D_1 \cap D_2 = \varnothing$ follows immediately from Definition 5(α) and from the fact that $\langle D_1^{T(n)} : n < \omega \rangle$ and $\langle D_2^{T(n)} : n < \omega \rangle$ are ascending sequences (by Lemma 6(2)). By Lemma 6(1) D_1 is dense in X, since every basic open set U_n contains points of $D_1^{T(n)} \subseteq D_1$. Similarly, by Lemma 6(7) also D_2 is dense in X.

(2) First let us prove that f maps $X - E_2$ into $X - E_1$. Let $x \in X - E_2$, then $f(x)$ is defined, as we saw above. Assume $f(x) \in E_1 = \bigcup_{n<\omega} E_1^{T(n)}$, then for some $n < \omega$, $f(x) \in E_1^{T(n)}$. $d(f_{T(n)}(x), E_1^{T(n)}) > 0$ since $E_1^{T(n)}$ is closed and $f_{T(n)}(x) \notin E_1^{T(n)}$

as $f_{T(n)}$ maps $X - E_2^{T(n)}$ onto $X - E_1^{T(n)}$, and $x \in X - E_2 \subseteq X - E_2^{T(n)}$. We shall prove soon that for all $m \geqq n$ we have

$$(*) \qquad d(f_{T(m)}(x), E_1^{T(n)}) \geqq \left(1 - \sum_{i=n}^{m-1} 2^{-(i+2)}\right) \cdot d(f_{T(n)}(x), E_1^{T(n)}).$$

Therefore $d(f_{T(m)}(x), E_1^{T(n)}) > \frac{1}{2} d(f_{T(n)}(x), E_1^{T(n)})$. Letting $m \to \infty$ we get

$$d(f(x), E_1^{T(n)}) \geqq \tfrac{1}{2} d(f_{T(n)}(x), E_1^{T(n)}) > 0,$$

contradicting $f(x) \in E_1^{T(n)}$. We prove now $(*)$ by induction on $m \geqq n$. For $m = n$, $(*)$ is trivial. We assume now $(*)$ for m and prove it for $m + 1$. By Lemma 6(4)

$$\begin{aligned} d(f_{T(m+1)}(x), E_1^{T(n)}) &\geqq d(f_{T(m)}(x), E_1^{T(n)}) - d(f_{T(m+1)}(x), f_{T(m)}(x)) \\ &\geqq d(f_{T(m)}(x), E_1^{T(n)}) - \varepsilon d(f_{T(m)}(x), E_1^{T(m)}) \\ &\geqq d(f_{T(m)}(x), E_1^{T(n)}) - \varepsilon d(f_{T(m)}(x), E_1^{T(n)}) \\ &= (1 - \varepsilon) d(f_{T(m)}(x), E_1^{T(n)}) \end{aligned}$$

and by $(*)$ for m we have

$$\begin{aligned} &\geqq (1 - \varepsilon)\left(1 - \sum_{i=n}^{m-1} 2^{-(i+2)}\right) \cdot d(f_{T(n)}(x), E_1^{T(n)}) \\ &\geqq \left(1 - \sum_{i=n}^{m} 2^{-(i+2)}\right) d(f_{T(n)}(x), E_1^{T(n)}). \end{aligned}$$

We saw already that $\langle f_{T(n)} : n < \omega \rangle$ is a uniformly convergent sequence of continuous functions, hence its limit f is also continuous. Similarly, by Lemma 6(5) also the sequence $\langle f_{T(n)}^{-1} : n < \omega \rangle$ is a uniformly convergent sequence of continuous functions; let $\bar{f}$ be its limit, then $\mathrm{Dom}(\bar{f}) = X - E_1$ and $\bar{f}$ is a continuous function. Exactly as we proved above for f we get that $\bar{f}$ maps $X - E_1$ into $X - E_2$. Since the sequences $\langle f_{T(n)} : n < \omega \rangle$ and $\langle f_{T(n)}^{-1} : n < \omega \rangle$ are uniformly convergent sequences of continuous functions we have

$$\bar{f} f = \lim_{n \to \infty} f_{T(n)}^{-1} \cdot \lim_{n \to \infty} f_{T(n)} = \lim_{n \to \infty} f_{T(n)}^{-1} f_{T(n)} = \lim_{i \to \infty} \text{identity} = \text{identity},$$

and similarly $f \bar{f} = \text{identity}$, hence $\bar{f} = f^{-1}$ and f is a homeomorphism of $X - E_2$ onto $X - E_1$.

(3) By Definition 3(γ), each $f_{T(n)}$ maps $D_1^{T(n)}$ onto $D_2^{T(n)}$ and $D_2^{T(n)} - E_2^{T(n)}$ onto $D_1^{T(n)} - E_1^{T(n)}$. By Lemma 6(2) and (3) the sequences $\langle D_1^{T(n)} : n < \omega \rangle$ and

$\langle D_2^{T(n)} - E_2^{T(n)} : n < \omega \rangle$ are increasing and for each $n, f_{T(n+1)}$ agrees with $f_{T(n)}$ on $D_1^{T(n)} \cup D_2^{T(n)}$. Therefore for each $m > n, f_{T(m)}$ agrees with $f_{T(n)}$ on $D_1^{T(n)} \cup D_2^{T(n)}$, hence also f agrees with $f_{T(n)}$ on $D_1^{T(n)} \cup D_2^{T(n)}$. Thus f maps $D_1^{T(n)}$ onto $D_2^{T(n)}$ and $D_2^{T(n)} - E_2^{T(n)}$ onto $D_1^{T(n)} - E_1^{T(n)}$. Our definition of D_1, D_2, E_1, E_2 easily implies that F maps D_1 onto D_2 and $D_2 - E_2$ onto $D_1 - E_1$.

(4) By (ε) $f_{T(n)}^{2k+1}(x) \neq x$ whenever it is defined. Since the sequence $\langle f_{T(n)} : n < \omega \rangle$ is a uniformly convergent sequence of continuous functions, we have $\lim_{n \to \infty} f_{T(n)}^{2k+1} = f^{2k+1}$. Thus we have to prove that $\lim_{n \to \infty} f_{T(n)}^{2k+1}(x) \neq x$ whenever it is defined. By Lemma 6(6) we have, for $m \geqq 2k+1$,

$$\begin{aligned} d(f_{T(m)}^{2k+1}(x), x) &\geqq \prod_{i=2k+1}^{m-1} (1 - 2^{-(i+2)}) d(f_{T(2k+1)}^{2k+1}(x), x) \\ &\geqq \prod_{i=2k+1}^{\infty} (1 - 2^{-(i+2)}) d(f_{T(2k+1)}^{2k+1})(x), x) \\ &\geqq \tfrac{1}{2} d(f_{T(2k+1)}^{2k+1}(x), x) \\ &> 0 \end{aligned}$$

Thus we have shown that for every $m \geqq 2k+1$, $d(f_{T(m)}^{2k+1}(x), x)$ is greater than some fixed positive number, hence also the limit $d(f^{2k+1}(x), x)$ is at least that number and $f^{2k+1}(x) \neq x$.

(6^{B}) Assume $V \neq \varnothing$ is an open set and $V - (D_1 \cup D_2) = A \cup B$ where A is meager and $|B| < 2^{\aleph_0}$. $D_1 \cup D_2$ is meager by Definition 3(α). Therefore, by the Baire category theorem and since X is complete and has no isolated points, $V - (D_1 \cup D_2 \cup A) \subseteq B$ included a perfect set, contradicting $|B| < 2^{\aleph_0}$.

(7) Given U let n be such that $U_n \subseteq U$. By Lemma 6(1) there is a sequence $\langle z_i : i < \omega \rangle$ of points of $D_1^{T(n)} \cap U \cap \operatorname{Dom} f_{T(n)}^{2m}$ which converges to a point of $D_1^{T(n)} \cap U_n \subseteq D_1 \cap U$ but where $\langle f_{T(n)}^{2m}(z_i) : i < \omega \rangle$ does not converge. Since $z_i \in D_1^{T(n)}$ we have, by Definition 5(γ), that all $f_{T(n)}^j(z_i)$, for $j \leqq 2n$ and $i < \omega$, are in $D_1^{T(n)} \cup D_2^{T(n)} \subseteq D_1 \cup D_2$. Since, as we saw above, f coincides with $f_{T(n)}$ in $D_1^{T(n)} \cup D_2^{T(n)}$ also f^{2m} coincides with $f_{T(n)}^{2m}$ on $D_1^{T(n)} \cup D_2^{T(n)}$ and thus the sequence $\langle f^{2m}(z_i) : i < \omega \rangle$ does not converge.

Lemma 7. *Assume that the assumptions of Lemma 4 hold with assumption (1), that $\langle X, d \rangle$ is a complete separable space without isolated points replaced by (1)′, $\langle X, d \rangle$ is a meager separable metric space without isolated points in which every nonvoid open set is of cardinality $2^{\aleph_0}$.*

Then there are f, D_1, D_2, E_1, E_2 which satisfy the hypotheses of Theorem 1(A).

PROOF. The proof is similar to that of Lemma 4, and we shall only point out the differences. First we shall prove Lemma 8 which is a strengthening of Lemma 6.

LEMMA 8. *For every approximation S, $\varepsilon > 0$, $k > 0$ and nonvoid open sets $U, R_1, \ldots, R_l$, where each R_j, $1 \leqq j \leqq l$ is such that $f_S^m(R_j) \cap f_S^n(R_j) = \emptyset$ for $m, n \in \mathbf{Z}$, $m \neq n$, and a nowhere dense set C, there is a nonvoid open set $V \subseteq U$ and an approximation T such that:*

(1), (2), (4)–(7), *as in Lemma* 6.

(3*) *f_T agrees with f_S and f_T^{-1} agrees with f_S^{-1} on $D_1^S \cup D_2^S \cup C$.*

(8) *For all $n \in \mathbf{Z}$, $d(f_T^n(V), \bigcup_{m \in \mathbf{Z}, m \neq n} f_T^m(V)) > 0$.*

(9) *If $C^* \subseteq C$, $C^* \cap (D_1^S \cup D_2^S) = \emptyset$ and C^* is closed under f_S and f_S^{-1} then $C^* \cap (D_1^T \cup D_2^T) = \emptyset$.*

(10) *We say that a set $R \subseteq X$ is* confined *by the transition from S to T iff for all $n \in \mathbf{Z}$, $f_T^n(R) \subseteq f_S^n(R)$. The given sets $R_1, \ldots, R_l$ and the set V are confined by the transition from S to T.*

PROOF OF LEMMA 8. The proof is like the proof of Lemma 6, with the following changes. We choose the set V so that V, V_1, V_2 and V_{-1} are disjoint from C and this takes care of (3*). The set V of the proof of Lemma 6 indeed satisfies (8). Since $C \cap V = \emptyset$, $C^* \subseteq C$ and C^* is closed under f_S and f_S^{-1}, we have also $C^* \cap V_n = \emptyset$ for all $n \in \mathbf{Z}$. Therefore $C^* \cap (D_i^T - D_i^S) = \emptyset$ for $i = 1, 2$ and since $C^* \cap D_i^S = \emptyset$ we have $C^* \cap D_i^T = \emptyset$, and (9) holds. To satisfy (10) we add to requirements (i)–(v) on V at the beginning of the proof of Lemma 6 the requirement:

(vi) For each $1 \leqq j \leqq l$ either for some $n \in Z$, $V \subseteq f_S^n(R_j)$ or else for every $n \in \mathbf{Z}$, $V \cap f_S^n(R_j) = \emptyset$.

To make sure that V satisfies (vi), for every $1 \leqq j \leqq l$ separately, we proceed as follows. If $V \cap f_S^n(R_j) = \emptyset$ for every $n \in \mathbf{Z}$ then V already satisfies (vi) for j, otherwise for some $n \in \mathbf{Z}$, $V \cap f_S^n(R_j) \neq \emptyset$. Since f_S is a homeomorphism of the open set $X - E_2$ onto the open set $X - E_1$ it preserves open sets. Similarly also f_S^{-1} preserves open sets. Thus $f_S^n(R_j)$ is an open set and we shrink V to $V \cap f_S^n(R_j)$ getting $V \subseteq f_S^n(R_j)$. To see that (10) holds let us recall that f_T coincides with f_S outside $V_{-1} \cup V_1$ and that, by the definition of f_T, $f_T(V_{-1}) = V_0 - E_1^T$ and $f_T(V_1) = V_2 - E_1^T$, f_T^{-1} coincides with f_S^{-1} outside $V_0 \cup V_2$, $f_T^{-1}(V_2) = V_1 - E_2^T$ and $f_T^{-1}(V_0) = V_{-1} - E_2^T$. Thus $f_T^n(V) \subseteq f_S^n(V)$ for every $n \in \mathbf{Z}$, $1 \leqq j \leqq l$ we have, by (vi), one of the following two cases. If $V \cap f_S^n(R_j) = \emptyset$ for every $n \in \mathbf{Z}$ then also $f_S^i(V) = V_i$ are disjoint from the sets $f_S^n(R_j)$, hence f_T and f_T^{-1} coincide with f_S and f_S^{-1}, respectively, in all sets

$f_n^S(R_j)$, hence $f_T^n(R_j) = f_S^n(R_j)$. The other case is that $V \subseteq f_S^n(R_j)$ for some $n \in \mathbf{Z}$. Since $f_S^m(R_j)$, for $m \in \mathbf{Z}$, are pairwise disjoint, f_T coincides with f_S outside $V_{-1} \cup V_1$, f_T^{-1} coincides with f_S^{-1} outside $V_0 \cup V_2$, $f_T(V_i) \subseteq f_S(V_i)$ for $i = -1, 1$ and $f_T^{-1}(V_i) \subseteq f_S^{-1}(V_i)$ for $i = 0, 2$, we have $f_T^m(R_j) \subseteq f_S^m(R_j)$ for $m \in \mathbf{Z}$.

Proof of Lemma 7 (continued). We define the sequence $\langle T(n) : n < \omega \rangle$, of approximations as in the proof of Lemma 4, with the following differences. For $n < \omega$, $\bar{V}^n$ is the set V of Lemma 8 obtained in the passage from $T(n)$ to $T(n+1)$. In this passage we take $l = n - 1$ and $R_j = \bar{V}^j$ for $j < n$. Since X is meager let $X = \bigcup_{n<\omega} C_n$, where for every $n < \omega$, $C_n \subseteq C_{n+1}$ and C_n is a closed nowhere dense set. When we construct the open set V, in the passage from $T(n)$ to $T(n+1)$ we add the following step. By our hypothesis, since V is open and nonvoid, $|V| = 2^{\aleph_0}$. Since $V = \bigcup_{j<\omega} V \cap C_j$ there is, by the Zermelo–König inequality, a $k < \omega$ such that $|V \cap C_k| = 2^{\aleph_0}$. $V \cap C_k \subseteq C_k$ is nowhere dense, and so is each set $f_S^j(V \cap C_k)$ for $j \in \mathbf{Z}$. Since, for every $j \in \mathbf{Z}$, $d(f_S^j(V), \bigcup_{n<\mathbf{Z}, n\neq j} f_S^n(V)) > 0$ also $\bar{C}_n = \bigcup_{j<\omega} f_S^j(V \cap C_k)$ is nowhere dense. $U_n \cap \bar{C}_n \supseteq V \cap C_k$ hence $|U_n \cap \bar{C}_n| = 2^{\aleph_0}$. We shrink V further so that $V \cap \bar{C}_n = \varnothing$. We take for C in the passage from $T(n)$ to $T(n+1)$ the set $C_n \cup \bar{C}_0 \cup \cdots \cup \bar{C}_n$. Notice that $\bar{C}_n$ is closed under f_S and f_S^{-1} and is disjoint from $D_1^S \cup D_2^S$ (since V is disjoint from $D_1^S \cup D_2^S$ and $D_1^S \cup D_2^S$ is closed under f_S and f_S^{-1}).

Let $x \in X - E_2$. For some $m < \omega$, $x \in C_m$ hence, by our choice of C and (3*), the sequence $\langle f_{T(n)}(x) : n < \omega \rangle$ is constant from $n = m$ onwards and we can define $f = \lim_{n\to\infty} f_{T(n)}$, as in the proof of Lemma 4. (1)–(4), (7) of Theorem 1(A) hold as in the proof of Lemma 4. To see that (5) holds let U_m be a basic open set. We saw above that $|U_m \cap \bar{C}_m| = 2^{\aleph_0}$, that $\bar{C}_m \cap (D_1^{T(m)} \cup D_2^{T(m)}) = \varnothing$ and that $\bar{C}_m$ is closed under $f_{T(m)}$ and $f_{T(m)}^{-1}$. Therefore, by (9) of Lemma 8 also $\bar{C}_m \cap (D_1^{T(n)} \cup D_2^{T(n)}) = \varnothing$ for all $n > m$ hence $\bar{C}_m \cap (D_1 \cup D_2) = \varnothing$. Thus $|U_m - D_1 \cup D_2| = 2^{\aleph_0}$. To prove (6), given a nonvoid open set U, let m be such that $U_m \subseteq U$, then $\bar{V}_m \subseteq U_m$ and by Lemma 8(8) for all $i \in \mathbf{Z}$,

$$d\left(f_{T(m)}^i(\bar{V}_m), \bigcup_{k\in\mathbf{Z}, k\neq i} f_{T(m)}^k(\bar{V}_m)\right) > 0.$$

By our choice of the R_j's, for each stage in the construction of the $T(n)$'s we have, by Lemma 8(10), that $\bar{V}_m$ is confined in each transition from $T(n)$ to $T(n+1)$, thus for each $i \in \mathbf{Z}$,

$$f^i_{T(m)}(\bar{V}_m) \supseteq f^i_{T(m+1)}(\bar{V}_m) \supseteq f^i_{T(m+2)}(\bar{V}_m) \supseteq \cdots.$$

Since f is the limit of the $f_{T(n)}$'s in the strong sense stated above we have $f^i_{T(m)}(\bar{V}_m) \supseteq f^i(\bar{V}_m)$. Therefore also $d(f^i(\bar{V}_m), \bigcup_{k \in \mathbf{Z}, k \neq i} f^k(\bar{V}_m)) > 0$.

Proof of the Main Theorem — Second Stage. If X is a nontrivial complete separable normed vector space over $\mathbf{Q}$, then it clearly satisfies the conditions of Lemma 4, where we take for F any nontrivial translation and for g an appropriate translation. By Theorem 1(B), this establishes part (4) of the Corollary to the main theorem. In particular the consequences of Lemma 4 hold for the case where X is the real line $\mathfrak{R}$. This is what was still needed in the first part of the proof of the Main Theorem to establish part (1) of the Main Theorem.

Lemma 7 and Theorem 1(A) imply similarly part (2) of the Main Theorem.

Proof of the Corollary to the Main Theorem. We have proved (4) directly, but it is also a particular case of (1). We shall establish (5) and (6) by showing that (5) implies (3), and (6) implies (1) or (3).

Theorem 9. *Let X be a normed vector space over $\mathbf{Q}$ with a nonvoid bounded clopen subset, then X has an autohomeomorphism of order 2 without fixed points.*

Proof. Let $z \in X$, $z \neq 0$. Since X has a nonvoid bounded clopen set V we can assume, without loss of generality, that $0 \in V$ and that V is of small enough diameter so that $V \cap (V+z) = \emptyset$. Clearly $\bigcup_{n \in \mathbf{Z}^+} nV = X$, where $\mathbf{Z}^+$ is the set of all positive integers. We define now for every $n \in \mathbf{Z}^+$ a bounded clopen set C_n and an automorphism f_n of C_n of order 2 with no fixed points, so that $C_{n+1} \supseteq C_n$ and $f_{n+1} \supseteq f_n$. For $n = 0$ we take $C_0 = \emptyset$, $f_0 = \emptyset$. Given C_n, since it is bounded there is an $m \in \mathbf{Z}^+$ such that $C_n \subseteq mV$. We set $C_{n+1} = mV \cup ((mV - C_n) + mz)$. Since $V \cap (V+z) = \emptyset$ also $mV \cap (mV + mz) = \emptyset$. We set

$$f_{n+1}(x) = \begin{cases} f_n(x), & x \in C_n, \\ x + mz, & x \in mV - C_n, \\ x - mz, & x \in (mV - C_n) + mz, \end{cases}$$

then clearly f_{n+1} is as required. Since $\bigcup_{n \in \mathbf{Z}^+} C_n = \bigcup_{m \in \mathbf{Z}^+} mV = X$, the jefunction $f = \bigcup_{n \in \mathbf{Z}^+} f_n$ is an autohomeomorphism of X of order 2 with no fixed points.

THEOREM 10. *Let X be a normed vector space without any complete direction and let f be a nontrivial autohomeomorphism of X which is the identity outside a bounded set, then X has an autohomeomorphism of order 2 without fixed points.*

PROOF. Since f is nontrivial there is an $x_0 \in X$ such that $f(x_0) \neq x_0$. Without loss of generality we have $f(0) \neq 0$, since otherwise we can replace f by f^* given by $f^*(x) = f(x + x_0) - x_0$, and f^* is clearly an autohomeomorphism of X which is the identity outside a bounded set and such that $f^*(0) \neq 0$. We can also assume that $\| f(0) \| = 1$, since otherwise we can replace the norm $\| x \|$ by the norm $\| x \|^*$ given by $\| x \|^* = \| x \| / \| f(0) \|$ without changing the topology of X. Let r be such that

(1) for all x such that $f(x) \neq x$, $\| x \|$, $\| x - f(0) \| < r$.

LEMMA 11. *For every $\varepsilon > 0$ there is a $\delta > 0$ such that for every real t, $0 < t \leqq 1$,*

(2) *if $\| x \| < \delta/t$ then $\| f(x) - f(0) \| < \varepsilon/t$,*

(3) *if $\| x - f(0) \| < \delta/t$ then $\| f^{-1}(x) \| < \varepsilon/t$.*

PROOF. We shall prove only (2), since (3) follows from (2) by substituting in (2) $x - f(0)$ for x and the function $f^{-1}(x + f(0)) - f(0)$, which satisfies the same hypotheses as f, for f.

Let $\varepsilon > 0$. Since f is continuous there is a $\delta_1 > 0$ such that, for $\| x \| < \delta_1$, $\| f(x) - f(0) \| < \varepsilon$. Let

$$\delta = \min\left(\frac{\delta_1 \varepsilon}{3r}, \frac{\varepsilon}{3}\right) > 0;$$

we shall see that δ satisfies (2). Let $0 < t \leqq 1$ and $\| x \| < \delta/t$. We shall distinguish the following three cases.

Case 1. $t \leqq \varepsilon/3r$ and $\| x \| \leqq r$. By our choice of r, $\| f(0) \| \leqq r$, and since $\| x \| \leqq r$ also $\| f(x) \| \leqq r$. Therefore $\| f(x) - f(0) \| \leqq \| f(x) \| + \| f(0) \| \leqq r + r = 2r$. In the present case $t \leqq \varepsilon/3r$ hence $2r < 3r \leqq \varepsilon/t$, thus $\| f(x) - f(0) \| < \varepsilon/t$.

Case 2. $t \leqq \varepsilon/3r$ and $\| x \| > r$. Since $\| x \| > r$, $f(x) = x$. Therefore

$$\| f(x) - f(0) \| \leqq \| f(x) \| + \| f(0) \| \leqq \| x \| + r < \delta/t + r.$$

By the definition of δ, $\delta/t \leqq \varepsilon/3t$; by the hypotheses of the present case $t \leqq \varepsilon/3r$ and hence $r \leqq \varepsilon/3t$; therefore

$$\| f(x) - f(0) \| < \delta/t + r \leqq \varepsilon/3t + \varepsilon/3t < \varepsilon/t.$$

Case 3. $t > \varepsilon/3r$. Therefore $1/t < 3r/\varepsilon$ and thus

$$\| x \| < \delta/t < \delta(3r/\varepsilon) \leqq \delta_1,$$

by the definition of δ. By the choice of δ_1, $\| f(x) - f(0) \| < \varepsilon \leqq \varepsilon/t$ (since $0 < t \leqq 1$).

PROOF OF THEOREM 10 (continued). We define a sequence k_n of positive integers by induction as follows: $k_0 = 1$; for $n \geqq 0$, k_{n+1} is such that $k_{n+1} > k_n$ and for all $0 < t \leqq 1$

(4) if $\| x \| < r/tk_{n+1}$ then $\| f(x) - f(0) \| < r/tk_n$;

(5) if $\| x - f(0) \| \leqq r/tk_{n+1}$ then $\| f^{-1}(x) \| < r/tk_n$.

The existence of k_{n+1} follows from Lemma 11.

We define, for $n \geqq 0$, $a_n = (r+1)/k_n$; clearly $\lim_{n\to\infty} a_n = 0$.

Since no direction in X is complete, there is a real $0 < q < 1$ such that $qf(0)$ is in the completion X^c of X but not in X. Let q_n be an ascending sequence of rationals such that $q_0 = 0$, $\lim_{n\to\infty} q_n = q$ and $q - q_n < 1/2k_{4n+1}$. Let $z_n = q_n f(0)$, then $z_0 = 0$ and $z = \lim_{n\to\infty} z_n = qf(0) \notin X$, $\| z_n - z_m \| = |q_n - q_m|$, $\| z - z_n \| = q - q_n$. Let $p_n = q_{n+1} - q_n$, then

(6) $p_n = q_{n+1} - q_n < q - q_n < 1/2k_{4n+1}$.

We have now

(7) $a_{n+1} > rp_n$,

since

$$p_n < \frac{1}{2k_{4n+1}} < \frac{1}{k_{n+1}} = \frac{1}{r} \cdot \frac{r}{k_{n+1}} < \frac{1}{r}\, a_{n+1},$$

and also

(8) $a_{n+1} > r/k_n + p_n$

since

$$p_n < \frac{1}{2k_{4n+1}} < \frac{1}{k_{n+1}} = a_{n+1} - \frac{r}{k_{n+1}} < a_{n+1} - \frac{r}{k_n}.$$

We shall construct a sequence g_n of autohomeomorphisms of X of order 2 such that

(9) z_n is the only fixed point of g_n,

(10) if $n > 0$ and $\| x - z_n \| \geqq a_n$, then $g_{n+1}(x) = g_n(x)$,

(11) for all $m \geqq n$, if $\| x - z_n \| < r/k_{m+3n}$ then $\| g_n(x) - z_n \| < r/k_m$.

We shall first show that the theorem follows from the existence of this sequence. Let $A_n = \{x \in X : \| x - z \| > a_n + q - q_n\}$; A_n is an open set and $A_n \subseteq A_{n+1}$. We have

$$
\begin{aligned}
x \in A_n \Rightarrow & \| x - z \| \geqq a_n + q - q_n \\
(12) \qquad & \Rightarrow \| x - z_n \| \geqq \| x - z \| - \| z - z_n \| \\
& \geqq a_n + q - q_n - (q - q_n) = a_n
\end{aligned}
$$

hence by (10)

(13) $x \in A_n \Rightarrow g_n(x) = g_{n+1}(x)$.

For every $x \in X$ let $n(x) = \min\{n : x \in A_n\}$; $n(x)$ exists since $\lim_{n \to \infty}(a_n + q - q_n) = 0$ and $z \notin X$. Let G be the function on X given by $G(x) = g_{n(x)}(x)$. By (13) G coincides with g_n on A_n, hence G is continuous on A_n, and since $X = \bigcup_{n<\omega} A_n$, G is continuous on X, and since the g_n's are of order 2 so is also G. Any fixed point of G has to be a fixed point of some g_n hence, by (9), it has to be z_n for some n, but, by (9), z_n is not the fixed point of any g_m with $m \neq n$, hence G has no fixed points and is as required by the theorem.

Now we define the sequence g_n. g_0 is defined by $g_0(x) = -x$. g_0 is clearly an autohomeomorphism of X of order 2 and it satisfies (9) and (11). In order to define g_{n+1} from g_n we define an autohomeomorphism h_n of X by

$$h_n(x) = p_n f\left(\frac{x - z_n}{p_n}\right) + z_n.$$

h_n is an autohomeomorphism, being a composition of autohomeomorphisms. By definition of the z_n's $h_n(z_n) = z_{n+1}$. We have

$$\| x - z_n \| \geqq rp_n \Rightarrow h_n(x) = x, \tag{14}$$

since

$$
\begin{aligned}
\| x - z_n \| \geqq rp_n \Rightarrow \left\| \frac{x - z_n}{p_n} \right\| \geqq r \Rightarrow f\left(\frac{x - z_n}{p_n}\right) = \frac{x - z_n}{p_n} \Rightarrow h_n(x) \\
= p_n \frac{x - z_n}{p_n} + z_n = x.
\end{aligned}
$$

We shall now prove

$$\| x - z_{n+1} \| \geqq rp_n \Rightarrow h_n^{-1}(x) = x. \tag{15}$$

Let $y = h_n^{-1}(x)$; it follows immediately from the definition of h_n that

$$y = p_n f^{-1}\left(\frac{x - z_n}{p_n}\right) + z_n.$$

By our assumption that $\| x - z_{n+1} \| \geqq rp_n$ and since $z_{n+1} - z_n = q_{n+1} f(0) - q_n f(0) = p_n f(0)$ we get

$$\left\| \frac{x - z_n}{p_n} - f(0) \right\| = \left\| \frac{x - z_n}{p_n} - \frac{z_{n+1} - z_n}{p_n} \right\| = \left\| \frac{x - z_{n+1}}{p_n} \right\| \geqq r.$$

By our choice of r, $\| x - f(0) \| \geqq r \Rightarrow f(x) = x$, hence also $f^{-1}(x) = x$ and we have

$$f^{-1}\left(\frac{x - z_n}{p_n}\right) = \frac{x - z_n}{p_n} \quad \text{and} \quad y = p_n \frac{x - z_n}{p_n} + z_n = x,$$

thus $h_n^{-1}(x) = x$, which is what we set out to show.

We show also that

$$\| x - z_n \| < \frac{r}{k_{m+1}} \Rightarrow \| h_n(x) - z_{n+1} \| < \frac{r}{k_m}, \tag{16}$$

$$\begin{aligned} \| h_n(x) - z_{n+1} \| &= \| h_n(x) - h_n(z_n) \| \\ &= \left\| p_n f\left(\frac{x - z_n}{p_n}\right) - p_n f(0) \right\| = p_n \left\| f\left(\frac{x - z_n}{p_n}\right) - f(0) \right\|. \end{aligned}$$

By the hypotheses of (16)

$$\left\| \frac{x - z_n}{p_n} \right\| < \frac{r}{p_n k_{m+1}}.$$

Substituting in (4) m, $(x - z_n)/p_n$ and p_n for n, x and t we get

$$\| h_n(x) - z_{n+1} \| = p_n \left\| f\left(\frac{x - z_n}{p_n}\right) - f(0) \right\| < p_n \frac{r}{p_n k_m} = \frac{r}{k_m}$$

so (16) holds. Similarly,

$$\| x - z_{n+1} \| < \frac{r}{k_{m+1}} \Rightarrow \| h_n^{-1}(x) - z_n \| < \frac{r}{k_m}. \tag{17}$$

Since

$$h_n^{-1}(x) = p_n f^{-1}\left(\frac{x - z_n}{p_n}\right) + z_n$$

we have

$$\| h_n^{-1}(x) - z_n \| = \left\| p_n f^{-1}\left(\frac{x - z_n}{p_n}\right) \right\| = p_n \left\| f^{-1}\left(\frac{x - z_n}{p_n}\right) \right\| .$$

Substituting in (5) m, $(x - z_n)/p_n$ and p_n for n, x and t, and recalling from the proof of (15) that

$$\frac{x - z_n}{p_n} - f(0) = \frac{x - z_{n+1}}{p_n}$$

we get

$$\| h_n^{-1}(x) - z_n \| = p_n \left\| f^{-1}\left(\frac{x - z_n}{p}\right) \right\| < p_n \frac{r}{p_n k_m} = \frac{r}{k_m},$$

thus (17) holds.

We define now $g_{n+1} = h_n g_n h_n^{-1}$. Since g_{n+1} is a conjugate of g_n in the autohomeomorphism group of X, g_{n+1} is also of order 2 and its single fixed point is $h_n(z_n) = z_{n+1}$. Thus all we have to do in order to finish the proof of Theorem 10 is to show that g_{n+1} satisfies (10) and (11). We prove now

(18) $$\| x - z_{n+1} \| \geqq a_{n+1} \Rightarrow \| g_n(x) - z_n \| \geqq r p_n .$$

Assume that the conclusion of (18) does not hold, i.e., if $y = g_n(x)$ then

$$\| y - z_n \| < r p_n \leqq r / k_{4n},$$

by (6). Therefore, by (11),

$$\| g_n(y) - z_n \| < r / k_n .$$

But since g_n is of order 2, $g_n(y) = x$, hence $\| x - z_n \| < r/k_n$, hence by (8)

$$\| x - z_{n+1} \| \leqq \| x - z_n \| + \| z_n - z_{n+1} \| \leqq \frac{r}{k_n} + p_n < a_{n+1},$$

contradicting the hypotheses of (18).

To prove (10) for $n + 1$ assume $\| x - z_{n+1} \| \geqq a_{n+1}$. By (18)

$$\| g_n(x) - z_n \| \geqq r p_n ,$$

and by (14)

$$h_n g_n(x) = g_n(x).$$

Also from $\| x - z_{n+1} \| \geqq a_{n+1}$, (7) and (15) we get $h_n^{-1}(x) = x$, hence

$$g_{n+1}(x) = h_n g_n h_n^{-1}(x) = h_n g_n(x) = g_n(x),$$

which establishes (10) for $n + 1$.

To prove (11) for $n + 1$ assume

$$\| x - z_{n+1} \| < r/k_{m+3(n+1)}.$$

By (17)

$$\| h_n^{-1}(x) - z_n \| < r/k_{m+3(n+1)-1}.$$

By (11) for n

$$\| g_n h_n^{-1}(x) - z_n \| \leqq \frac{r}{k_{m+3(n+1)-1-3n}} = \frac{r}{k_{m+2}},$$

hence by (16)

$$\| h_n g_n h_n^{-1}(x) - z_{n+1} \| < \frac{r}{k_{2_m+2-1}} < \frac{r}{k_{m+1}},$$

i.e., $\| g_{n+1}(x) - z_{n+1} \| < r/k_{m+1}$.

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